Ring Theory 2

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With $1 \neq 0$ as identity in R, we have the units form a group of multiplication U.

$a \in R$, $\exists\, u \in U \subseteq R$ $ua = au = 1$ $\Rightarrow u$ is a unit in $R \supseteq U$

$1 \in R$, $\exists\, 1 \in U \subseteq R$ $1 \cdot 1 = 1 \cdot 1 = 1$ $\Rightarrow 1 \in U$

$\Rightarrow$ U is a group of units

•> A field F is a commutative ring with identity $1 \neq 0$ where we have every non-zero element is a unit

$a \in F \Rightarrow a a^{-1} = 1 = a^{-1} a$

$a^{-1} \in F$ if $a \neq 0$

•> R is a ring of all function in the closed interval $[0,1] \to \mathbb{R}$.

zero divisor $\Rightarrow$ Suppose $f \in R$ is any arbitrary function

If $gf = fg = 0$

where c a constant is not zero

$$g = \begin{cases} c & \text{if } f(x) = 0 \\ 0 & \text{if } f(x) \neq 0 \end{cases}$$

To g to be a zero-divisor f must be non-zero

$gf =$ either $c \cdot 0$ or $0 \cdot f = 0$

So g is the zero divisor of f

Unit $\Rightarrow$ $1 \in R$ and $1 \neq 0$. $\mathrm{Id} : [0,1] \to \mathbb{R}$, $x \to x$ is 1 here

$f \in R$ is any arbitrary function

Suppose $gf = fg = 1$, then,
$$g = \begin{cases} \frac{1}{f} & \text{if } f \neq 0 \ \forall x \in [0,1] \\ \text{undefined} & \text{otherwise} \end{cases}$$

•> An example of ring which doesn't have any identity.

$\Rightarrow$ $n\mathbb{Z}$ for $n \geq 2$

•> If R_1 subring of R_2 and R_2 is a subring of R_3

• If R_1 subring of R_2 and R_2 is a subring of R_3

$\Rightarrow$ $a \in R_1 \Rightarrow a \in R_2 \Rightarrow a \in R_3$

$a, b \in R_1 \Rightarrow a - b \in R_1$ & $ab \in R_1$

• $\mathbb{Q}$ is the set of all rationals. $(\mathbb{Q}, +, \cdot)$ is a ring.

• Set of all non-negative rationals $\Rightarrow$ not a ring

• " " " squares of " . $\Rightarrow$ not a ring

• " " " rationals with odd numerator $\Rightarrow$ not a ring

• " " " irrationals $\Rightarrow$ not ring

• " " " rationals with odd denominator $\Rightarrow$ $\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \rightarrow$ odd

$\frac{a}{b} + \left(\frac{-a}{b}\right) = 0$

$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd} \rightarrow$ odd

• " " " rationals with even denominator $\Rightarrow$ $\frac{1}{2} - \left(-\frac{1}{2}\right) = 1 = \frac{1}{1} \rightarrow$ odd

not closed under addition

$\Rightarrow$ not a ring

• Any finite integral domain is a field.

$\Rightarrow$

• R is a ring of functions from $[0,1] \rightarrow \mathbb{R}$

• the set S_1 of all $f(x)$ such that $f(x) = 0 \ \forall x \in \mathbb{Q} \cap [0,1]$

$\Rightarrow$ $f, g \in S_1$, $(f - g)(x) = f(x) - g(x) = 0 \ \forall x \in \mathbb{Q} \cap [0,1]$

$f(x) g(x) = 0 \ \forall x \in \mathbb{Q} \cap [0,1]$

• the set of all polynomial functions S_2

$\Rightarrow$ $f, g \in S_2$

$f = a_n x^n + \cdots + a_1 x + a_0$

$g = b_n x^n + \cdots + b_1 x + b_0$

$f - g = (a_n - b_n) x^n + \cdots + (a_0 - b_0) \in S_2$

$f g = a_n b_n x^{2n} + \cdots + a_0 b_0 \in S_2$